# White Paper on IPAM Long Program "Quantitative Linear Algebra"

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## What is Quantitative Linear Algebra?

The purpose of the program was to bring together topics from a number of important directions, including discrepancy theory, spectral graph theory, random matrices, geometric group theory, ergodic theory, von Neumann algebras, as well as specific research directions such as the Kadison-Singer problem, the Connes embedding conjecture and the Grothendieck inequality.

A very important aspect of the program is its aim to deepen the links between research communities working on some infinite-dimensional analysis problems that occur in geometric group theory, ergodic theory, von Neumann algebras; and some quantitative finite-dimensional ones that occur in spectral graph theory, random matrices, combinatorial optimization, and the Kadison-Singer problem.

The main events of the program were as follows:

- Tutorial lectures
- Workshop 1: Expected characteristic polynomials techniques and applications
- Workshop 2: Approximation properties in operator algebras and ergodic theory
- Workshop 3: Random matrices and free probability theory
- Culminating workshop at Lake Arrowhead
- Weekly seminars in between the workshops organized by the participants

**Random Matrix Theory**

Let $A$ be a self-adjoint matrix of size $N \times N$. This matrix can be fully described by giving the set of its eigenvalues $\lambda_1, \ldots, \lambda_N$ and of the associated eigenvectors. Of these, the eigenvalues are similarity invariants of the matrix, and so sometimes one is interested in the eigenvalues of the matrix only. There are several ways to encode the eigenvalues of the matrix. One is the so-called empirical distribution, which is the probability measure $\frac{1}{N}\sum_{i=1}^{N} \delta_{\lambda_i}$. In turn, this probability measure can be encoded by its moments, which can be rewritten as traces of powers of the matrix: $Tr(A^p)$, $p = 1,2, \ldots$. Alternatively, the measure can be encoded by its Stieltjes transform, i.e. the trace of the resolvent $Tr[(A - z)^{-1}]$, with $z$ lying in the upper half plane. Entries of the resolvent also provide information on the eigenvectors, in particular allowing one to distinguish whether they are *localized* (essentially supported on only a few coordinates) or *delocalized*.

In many applications, the matrix $A$ is random; in this case the eigenvalues of the matrix are random as well. **Random matrix theory** deals with questions of distribution of these eigenvalues, as well as asymptotic behavior of such eigenvalues. For example, suppose that the matrix $A$ has random entries which are complex Gaussian random variables (maximally independent subject to the constraint that $A$ is self-adjoint) and each has variance $1/N$ (so each row or each column has variance 1). In that case, a lot of detail is known about the behavior of eigenvalues and eigenvectors of the matrix $A$, mainly through the analysis of the average value and of fluctuations of its empirical distribution:

- With high probability, the eigenvalues belong to the interval $[-2,2]$
- The expected number of eigenvalues in the interval $[a, b]$ is given by the semicircle law, the ratio of the area under the semicircle of radius $2$ above the interval to the area of the semicircle of that radius;
- If $f$ is a function so that the expected value of $f(\lambda_i)$ is $i$, then the spacings $f(\lambda_i) - f(\lambda_{i+1})$ are distributed according to the "sine kernel" distribution
- The maximal and minimal eigenvalues are distributed according to the Tracy-Widom distribution.
- Regarding eigenvectors: the eigenbasis is distributed according to Haar measure on the unitary group. In particular, with high probability all of the eigenvectors are delocalized: their mass is spread more or less uniformly over all $n$coordinates.

This so-called **Gaussian Unitary Ensemble** (GUE) model is perhaps the best-understood one of all. Moreover, many of these results generalize to matrices with i.i.d. entries which are not

Gaussian but which have sufficiently many finite moments. However, there are a number of questions that go beyond this paradigm, often motivated by various applications:

a) Perhaps most notably, the above paradigm omits **non-Hermitian** random matrices, which present unique challenges for analysis. In particular, the spectrum of such matrices can be highly unstable under small perturbations, and the spectrum is not characterized by its moments. Nevertheless, non-Hermitian models have been an active area of interest in free probability, mathematical physics, and applications for modeling stability of dynamical systems (e.g. food webs, neural networks). The need to establish quantitative stability of the spectrum interfaces with the subject of **geometric functional analysis**.
b) **Quantitative theory:** Estimates for the norm of a random matrix and the norm of its inverse are useful for problems in signal processing, in addition to the above-mentioned applications to spectral stability for non-Hermitian models. There is also interest in obtaining such estimates under additional structural assumptions on the matrix distribution.
c) **Adjacency matrices** of random graphs: here spectral information about the matrix gives structural information about the graph, including questions such as its expansion properties. While Erdös–Rényi graphs with constant edge density fit easily into the iid paradigm, new approaches are needed to understand sparse graphs, or graphs whose adjacency matrices have dependent entries (e.g. d-regular graphs). One can also consider directed graphs, which yield non-Hermitian matrix models. There are also important questions concerning eigenvectors, such as the number of nodal domains, which have applications in computer science.
d) **Sparse random matrices**: In many applications, an appropriate model would be that of matrices whose entries are mostly zero; there is also the closely related models of matrices with heavy-tailed entries. Extending known techniques to the sparse case often requires near-optimal quantitative estimates.
e) **Combinations of several random and non-random matrices**: e.g., $A = X + Y$ where $X$ is random ("noise") and $Y$ is deterministic ("signal"); here the question is deducing properties of $Y$knowing spectral properties of $A$.
f) **Structured matrices**, in which one assumes that certain entries have different variance profile, e.g. "band matrices" where the entries are random but the $(i, j)$-th entry is zero if $|i - j| > \phi(N)$ for some function $\phi$. Work on such models is inspired by the *Anderson localization phenomenon* in mathematical physics. In particular, it is expected that for $\phi(N) >> \sqrt{N}$ the eigenvalues and eigenvectors of such models behave as they do for GUE, whereas for $\phi(N) << \sqrt{N}$there is a transition to Poisson statistics for eigenvalues and delocalization of eigenvectors. This transition models the metal/insulator transition in disordered materials.
g) **Unitarily invariant ensembles**, in which rather than assuming independence condition on entries, one assumes that the matrices are samples from a unitarily invariant

distribution of the form $exp(-\beta NTr(V(A))$ for some function $V$ (note that quadratic $V$ correspond to the GUE case).

h) **Large deviations**: For the GUE one can use the explicit formula for the joint density of eigenvalues to establish a large deviations principle for the empirical spectral distribution – that is, fine estimates on the probability that the spectrum is significantly different from its “equilibrium” semicircular distribution – as was done by Ben Arous and Guionnet in the early 2000s. Accomplishing the same for matrices with non-Gaussian entries is a long-standing open problem. One can similarly ask for large deviation principles for the largest eigenvalues, which could be useful for community detection in random graphs, or for the rigorous study of complex energy landscapes (which arise in the theory of deep neural networks).

i) Other statistics of the characteristic polynomial: Apart from its zeros, there is interest in other properties of the characteristic polynomials of random matrices, which fall into the paradigm of **logarithmically correlated fields**. Much interest is motivated by a well-known analogy between the characteristic polynomial of a Haar unitary matrix and statistical behavior of the Riemann zeta function. This has provided a useful tool for heuristic understanding of conjectures in analytic number theory.

These are just some of the motivating problems that were discussed at **Workshop III** devoted to random matrix techniques. There has been enormous progress on understanding some of these additional directions.

Some relied on updates to the **moment method** in which careful analysis of moments of the matrix enables control of its eigenvalues. A powerful variant of the moment method involves **analysis of non-backtracking paths**, interpreting the matrix as a random walk operator on a certain random graph and understanding the combinatorics of certain paths. This method has been useful for the study of the spectral gap and community detection for sparse random graphs. In particular, Florent Benaych-Georges talked about recent advances on the large eigenvalues of sparse random graphs using this method.

Others involve the use of super-fast **relaxation to GUE statistics**, a method pioneered by H.T. Yau, L. Erdös and collaborators, in which it is proved that a small GUE perturbation of many matrices immediately has GUE statistics, an observation that can be used to relate statistics of GUE matrices and of many other random matrices. At the workshop, Yau and his students J. Huang and B. Landon reported on recent upgrades to the relaxation method that yield fine information on the eigenvectors and spectrum for random band matrices and sparse random graphs. Regarding the universality theory for sparse random graphs, there were also talks of Bauerschmidt and Dumitriu on local laws for bounded-degree regular graphs and stochastic block models, respectively.

Progress on the universality theory of non-Hermitian random matrices has surged over the last decade, since work of Tao–Vu, Götze and A. Tikhomirov and others on the circular law describing the limiting spectral distribution at global scale. In particular, there has been progress extending the circular law class to include sparse directed graph ensembles by workshop participants Rudelson, Basak, Zeitouni, Cook, Tikhomirov and coauthors. At the workshop, Mark Rudelson and Konstantin Tikhomirov reported on recent progress with Anirban Basak extending the circular law to very sparse random matrices, which required new quantitative methods from geometric functional analysis. In a completely different direction, Ofer Zeitouni discussed banded non-Hermitian random matrices with bounded band width, which can display a variety of singular phenomena not seen in the circular law paradigm. His results with Basak and Paquette provoked a wide range of questions and provide fertile ground for future research.

There was a lot of interest among workshop participants in the properties of eigenvectors. Marc Potters discussed their importance in the analysis of data matrices, where tools from free probability have proved useful. At the end of the workshop, Mark Rudelson and Han Huang announced new results on the number of **nodal domains** for eigenvectors in Erdös–Rényi graphs. Their work benefited from the interaction between experts on universality theory and geometric functional analysis methods.

The theory of the supremum of log-correlated Gaussian fields was extended to include the logarithm of characteristic polynomials of random matrices, allowing to prove their convergence and fluctuations. In particular, precise results on the value of the supremum for the CUE field were obtained in the past couple of years, which led to some analogous results for the Riemann zeta function. Participants Cook and Zeitouni showed that some of the same behavior extends to characteristic polynomial of a random permutation matrix. This sparked some discussion of to what extent we should expect universality to hold for the behavior of the maximum of characteristic polynomials for various ensembles. The theory of traffics was adapted to show convergence of conditional expectations of powers of certain random matrices onto algebras of diagonal matrices.

It is expected that in the large deviations regime for various random matrix statistics we should see a variety of **non-universal** behaviors. Following work of Bordenave and Chafaï, Fanny Augeri has established large deviations principles for matrices whose entries have stretched exponential tails, which proved to have very different behavior from the GUE.

**The Method of Characteristic Polynomials.**

In their breakthrough work, Marcus, Spielman and Srivastava have replaced the empirical spectral distribution by another observable, the characteristic polynomial. Rather than looking at the questions of the average distribution of empirical measures, they were able to use interlacing polynomial techniques to control the **expected characteristic polynomials** of certain random matrices.

These polynomials are of interest from several different points of view, which were explored in the first workshop.

(a) The expected polynomials of certain **random graph** models correspond to well-studied objects in statistical physics, most notably the monomer-dimer partition function (**matching polynomial**). A few years ago this connection was used to prove the following conjecture of Bilu and Linial: every bipartite regular graph admits a 2-cover whose new Laplacian eigenvalues are bounded by the spectral radius of its universal cover, which immediately implies the existence of infinite families of bipartite **Ramanujan graphs**. Doron Puder presented his recent work with Hall and Sawin which shows using representation theory that the same kind of result is true in much greater generality, in particular that every graph has a $k$-cover with the same property, for every integer $k$ greater than 1.

(b) An alternate approach to constructing Ramanujan graphs is to directly study the expected characteristic polynomials of random graphs, rather than going via their covers. In this case, it turns out that the relevant polynomials come equipped with a natural notion of convolution that mimics the operation of **free convolution** in free probability theory, which takes place in the infinite-dimensional setting. Several talks in the workshop explored this finite-infinite connection; in particular Vadim Gorin explained how the zeros of expected characteristic polynomials may be viewed as a $\beta \to \infty$ limit of certain **Coulomb gas** models, and Octavio Arizmendi discussed how taking appropriate $n \to \infty$ limits of the polynomial convolutions yields the usual free convolution, showing that it is in some sense a discretization of (a part of) the infinite theory. The extent of the analogy between the finite and infinite convolutions is still not understood and remains a topic of investigation.

(c) In addition to combinatorics, expected characteristic polynomials can be used to study several random matrices arising in **functional analysis** problems. For instance, Pierre Youssef explained how such methods give improved bounds in Bourgain and Tzafriri's restricted invertibility theorem, which is a quantitative version of the fact that the number of linearly independent columns of a matrix is equal to its rank, and Mohan Ravichandran used interlacing polynomials to give a quantitatively sharp solution of the paving problem, which is a simple matrix analytic statement known to be equivalent to the Kadison-Singer problem.

(d) Dyson and Montgomery famously noticed in the 70's that the **zeros of the Riemann zeta function** look statistically similar to the eigenvalues of certain random matrices. Workshop one included two lectures following this theme, in particular connecting expected characteristic polynomials to analytic number theory. Alex Gamburd explained how the expected characteristic polynomials of random unitary matrices are relevant to computing these statistics, and presented results giving combinatorial interpretations of some of them. Terry Tao presented his recent proof with Rodgers of Newman's conjecture, which states that the zeros of the zeta function are "barely on the critical line", in that evolving (an appropriate renormalization of) the function by a reverse heat flow would immediately produce zeros that are not on the line. The connection is that the flow on the zeros is precisely the same as what one gets when computing the expected characteristic polynomial of a random Gaussian perturbation of a fixed matrix.

(e) Many of the tools used to analyze expected characteristic polynomials come from the theory of **stable polynomials**, which are a multivariate generalization of real-rooted polynomials. It turns out that several interesting **counting problems**, for instance computing the **permanent** of a matrix, can be reduced to understanding how the coefficients of stable polynomials behave under certain linear transformations. The workshop concluded with several talks exploring the interface between this theory and algorithmic problems in computer science; for instance, Barvinok and Saberi talked about computing the permanent of certain restricted classes of matrices, and Anari announced an exciting new result on approximately counting the number of bases of any matroid, building on work of Huh, Adiprasito, and Katz. These techniques are different from those used in analyzing expected characteristic polynomials, but share some common phenomenology.

**Finite-dimensional Approximation of Infinite-dimensional Systems**

A *tracial von Neumann algebra* is an infinite-dimensional generalization of the matrix algebras $M_N(C)$ with the normalized trace $(1/N)Tr$. More precisely, it is an algebra $M$ of operators on a Hilbert space, closed under adjoints and weak limits, equipped with the operator norm and with a linear functional $\tau: M \to C$called the trace satisfying $\tau(ab) = \tau(ba)$, $\tau(I) = 1$, and $\tau(a^*a) \geq 0$.

Tracial von Neumann algebras often arise as large-$N$limits of random matrix models, while other examples come from discrete groups. For a group $G$, one defines the group algebra as the vector space with basis $G$ and multiplication which linearly extends the group multiplication. The linear map given by $\tau(g) = \delta_{g=e}$ defines a trace on the completion of this algebra.

Free probability studies how well finite-dimensional matrix models can be approximated by the infinite-dimensional theory of tracial von Neumann algebras. In the opposite direction, the **Connes embedding question** asks whether every tracial von Neumann algebra can be simulated by finite matrices in following sense. Given an abstract tracial von Neumann algebra $(M, \tau)$ and self-adjoint elements $a_1, \dots, a_k$, is it true that for any $\epsilon > 0$ and any $d$, that there exist some self-adjoint matrices $A_1, \dots, A_k$of some size $N \times N$ with the property that $|(1/N)Tr(p(A_1, \dots, A_k)) - \tau(p(a_1, \dots, a_k))| < \epsilon$ for any monomial $p$ of degree at most $d$.

This question is open even for elements of group algebras. Moreover, in the group context, one can ask the related question whether every group can be simulated by finite permutation matrices (rather than simply unitary matrices as in Connes embedding). More precisely, let $G$ be a discrete group generated by $g_1, \dots, g_k, {g_1}^{-1}, \dots, {g_k}^{-1}$ . We say that $G$is *sofic* if for every $\epsilon > 0$ and $d > 0$, there exist permutation matrices $G_1, \dots, G_k$ of some size $N \times N$ with the property that for any monomial $p$ of degree at most $d$,$(1/N)\#\{fixed\ points\ of\ p(G_1, \dots, G_k, {G_1}^{-1}, \dots, {G_k}^{-1})\}$ is within $\epsilon$ of 0 or 1 depending on whether $p(g_1, \dots, g_k, {g_1}^{-1}, \dots, {g_k}^{-1})$ is trivial or not in $G$.

It is not hard to see that If $G$is amenable, residually finite, or more generally residually amenable, then $G$is sofic, and consequently the Connes embedding question has a positive answer for the group algebra of $G$. However, it is unknown whether all groups are sofic (this is called the **sofic group question**), and there are even some concrete groups whose soficity is unknown.

There are quantitative versions of the Connes embedding and sofic group questions. For a tracial von Neumann algebra $M$generated by $X_1, \dots, X_n$, Voiculescu's free entropy measures the amount of matrix approximations to $X_1, \dots, X_n$, and is closely related to large deviations in random matrix theory.

A similar idea lies behind Bowen's notion of sofic entropy for probability-preserving actions of a sofic group *G*. To define this, one fixes a tuple of permutations of a finite set {*1,2,...,N*} which approximate the relations of the group, and then counts the number of colorings of the set {*1,2,...,N*} which approximate the given action in combination with those fixed permutations. Sofic entropy provides an entropy-like invariant for actions of many non-amenable groups, including free groups. It promises a vast generalization of several older directions of research in ergodic theory, and has uncovered novel phenomena that cannot arise in the classical setting of amenable-group actions. Some of these were discussed at workshop II. For instance, the sets of finitary colorings which approximate a given action also admit natural measures of 'connectedness', which may be non-trivial for actions of non-amenable groups. One highly 'disconnected' example has recently been shown by Bowen to violate the 'weak Pinsker property', one of the main structures present in all actions of amenable groups.

Among other examples, S. Popa spoke on a cohomological interpretation of a number of questions -- including the Connes embedding question -- in terms of a certain non-abelian cohomology associated to groups. Thanks to his work, there emerged a very interesting class of groups with vanishing cohomology (a class that includes free groups, but which may be as large as groups measurably equivalent to free groups).

A reformulation of the Connes embedding question is the question of embeddability of an arbitrary abstract von Neumann algebra into an ultraproduct of matrix algebras. Such considerations turn out to be connected to **model theory** in mathematical logic, and there were several talks elucidating this connection.

The Connes embedding question made a remarkable appearance recently in quantum information theory, where it was related to approximation of sets of quantum correlations (abstract quantum systems vs. finite-dimensional quantum systems). A talk by Thomas Vidick and subsequent discussion with W. Slofstra were very inspirational, and have led M. Musat and M. Rørdam to the solution in the negative of the question if factorizable quantum channels (completely positive maps) can always be represented in finite dimensions.

**Other Approximation Properties in Ergodic Theory and von Neumann Algebras**

These "modeling" questions are in turn related to others that concern purely infinite-dimensional objects: Banach spaces, von Neumann algebras, etc.

This field of von Neumann algebras has seen a number of dramatic recent advances thanks in many ways to connections with free probability, geometric group theory, and the emergence of Popa's deformation-rigidity theory.

A number of results on classification of von Neumann algebras have their roots in finite-dimensional questions concerning approximation properties of these algebras. One such (anti-) approximation property is Kazhdan's property (T). A remarkable result of Ozawa et al states that the group of automorphisms of the free group on 5 generators has this property, a proof that (through very clever analysis) can tested on a computer by finding a sufficient condition that amounts to solving a certain finite-dimensional optimization problem. Such computer-assisted proofs have led to a number of dramatic improvements in the known Kazhdan constants for certain groups. In combination with work of Lubotzky and Pak, property (T) for the automorphism group of the free group on $k$ generators shows that the $k$-generator replacement graphs on arbitrary groups are expanders. This leads to the product replacement algorithm to generate uniformly distributed random elements in a given finite group. A new result by I. Pak proved during the program shows that the result of Ozawa et al for the automorphism group of 5-generator free group is sufficient to yield the expander property for $k$-generator replacement graphs for any fixed $k \geq 5$.

Kazhdan's property (T) and, more generally, spectral gap properties play a key role in Popa's deformation/rigidity theory. The basic and wide open question here is to which extent groups, equivalence relations or other discrete structures can be recovered from their ambient and highly malleable von Neumann algebra. Several talks focused on aspects of this question and its connections to lattices in Lie groups, $L^2$-invariants, measurable group theory and specific families of groups given by free product type constructions.

A famous application of random matrix techniques is the Johnson–Lindenstrauss lemma, which states that it is possible to project a high-dimensional space into low-dimensional space with bounded distortion. This results in **dimension reduction**, and is very desirable in applications. On the other side, there were talks by A. Naor and G. Schechtman about situations where such reduction is not possible.

Vaughan Jones gave the **Green Family Lecture Series**. His research talk (part of the second workshop) concentrated on a tantalizing way to discretize symmetries involved in conformal field symmetry using the Thompson group. Motivated by this, he discussed a new and remarkable family of representations of this group.

**Activities between Workshops.**

During the weeks with no workshops scheduled, the participants organized several seminar series and a series of open problem discussions. A weekly seminar series focused on random matrices and their application. In addition, there was a general seminar series which included a number of talks by junior participants, often multiple talks in the same week. These talks covered a broad range of topics such as operator algebras and free probability. A third seminar series focused on expected characteristic polynomials, and explored two main themes (1) connections between expected characteristic polynomials of sums of unitarily invariant matrices and free probability (2) the limiting behavior of the roots of expected characteristic polynomials and their relation to equilibrium measures in potential theory. Interest in (2) was triggered during the semester by conversations between participants from different communities e.g. Potters, Marcus, Srivastava, Grebinski, Jekel.

The open problem session has produced a number of highly stimulating questions. For example, one of the questions dealt with quantifying impossibility of satisfying the Heisenberg commutation relation $[x, y] = iI$ by bounded operators $x, y$. An old result of Popa shows that if $||[x, y] - I|| < \epsilon$, then $||x|| \cdot ||y|| \geq \frac{1}{2} \, log \, \frac{1}{\epsilon}$. A recent result of Tao improves the Brown-Pearcy lower estimate, showing that there are examples of $x, y$ with $||[x, y] - I|| < \epsilon$ but $||x|| \cdot ||y|| = O(log^{16} \, (\frac{1}{\epsilon}))$.

**Impact of the Program, Future Research Directions, and Connections.**

A number of junior participants mentioned that the program gave them ample opportunities to discuss their research with senior mathematicians, especially during the less busy weeks when there were no workshops. The tutorials week was an excellent preparation for the various topics discussed during the program, and the program itself was very useful in communicating the broad perspectives of research touched upon during the program. We also expect that the high quality videos produced during the tutorials week will be a valuable resource for researchers wanting to enter these fields in the future.

There have been several suggestions for possible follow-up topics. One involved applications of random matrix theory and spectral graph theory, including further discussion of applications to computer science, as well as applications to disciplines such as mathematical finance and experimental disciplines, such as computational biology, neurobiology, population statistics, physics, etc. One of the challenges here is to formulate better models for the noise that is inherent in such data, and to use random matrix theory and free probability theory to find ways to denoise the data. This interesting topic would seem to be worthy of a **separate IPAM workshop** (that could be titled "Beyond the Marchenko-Pastur Law").

One of the very exciting developments during the program is a **new book by M. Potters**, titled "A First Course in Random Matrix Theory". This book is aimed at a broad audience of physicists, engineers and computer sciences, has been influenced by many of the talks and discussions during the program.

Several of the open questions explored during the program have arisen from a synthesis of the various research directions presented in the different workshops. T. Austin spoke in the second workshop on his recent work establishing the weak Pinsker property for amenable groups: every measure-preserving action of any amenable group can be split as a direct product of a Bernoulli system and a system of arbitrarily small entropy. On other hand, L. Bowen explained a construction showing that the weak Pinsker property fails for non-abelian free groups. This leads to the intriguing question as to whether having the weak Pinsker property characterizes amenability.

A recent result of M. Musat and M. Rørdam giving the solution in the negative of the question if factorizable quantum channels (completely positive maps) can always be represented in finite dimensions was a natural outgrowth of discussions with W. Slofstra on quantum information theory.

A recent result by I. Pak, in combination with the result of Ozawa et al on property (T) for the automorphism of the free group on 5 generators showed that the product replacement graphs on arbitrary $k$-generator groups, $k \geq 5$, are expanders, explaining the efficiency of the product replacement algorithm.

An open question of Ben Hayes connects random matrix theory with with tensor norms in operator algebras. It concerns the limiting behavior of the random matrix model $(A_i, B_i : i = 1, \ldots, k)$ where $A_i, B_i$ are iid unitary random matrices, but $A_i$ act on matrices by left multiplication while $B_i$ act by right multiplication. The question is whether, for any polynomial $p$

in $2k$ variables (where the first $k$ and the last $k$ variables commute), one has that almost surely $||p(A_1, \ldots, A_k, B_1, \ldots, B_k)|| \rightarrow ||p(u_1 \otimes 1, \ldots, u_k \otimes 1, 1 \otimes u_1, \ldots, 1 \otimes u_k||$ where $u_j$ are free generators of the reduced C*-algebra of the free group and the norm on the right-hand side is the reduced C*-tensor norm. If there are no matrices $B$, this reduces to a well-known result of Haagerup and Thorbjørnsen. Solving this question would have immediate implications for von Neumann algebra theory.

Discussions during the workshop ignited research collaborations between many participants, for example Marcus, Anari, and Oveis Gharan and Srivastava and Meka, on outstanding problems concerning expected characteristic polynomials including: finding $l_1$ analogues of the proof of Weaver's conjecture (which is equivalent to the Kadison-Singer problem), which would have major graph theoretic implications; finding a polynomial time algorithm to produce the matrix pavings guaranteed by the currently nonconstructive theorem of Marcus, Spielman, and Srivastava. Between workshops there was an ongoing conversation including participants Potters, Srivastava, Shlyakhtenko, and several graduate students, aimed at understanding whether results in Coulomb gas theory could be used to explain in a precise way why the method of interlacing polynomials seems to always produce results that are sharp.