Singular Perturbations in Stochastic Control and Hamilton-Jacobi-Bellman Equation

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joint work with Martino Bardi

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Problem

Goal: study the limit[1] as $\varepsilon \to \mathbf{0}$, of the system

$$
\begin{aligned}
dX_t &= \boldsymbol{f}(X_t, Y_t, \boldsymbol{u_t})\,dt + \sqrt{2}\boldsymbol{\sigma}^{\varepsilon}(X_t, Y_t, \boldsymbol{u_t})\,dW_t,\ X_0 = x \in \mathbb{R}^n \\
dY_t &= \frac{1}{\varepsilon}\boldsymbol{b}(X_t, Y_t)\,dt + \sqrt{\frac{2}{\varepsilon}}\boldsymbol{\varrho}(X_t, Y_t)\,dW_t, \quad Y_0 = y \in \mathbb{R}^m
\end{aligned}
\qquad (\mathrm{SDE}(\tfrac{1}{\varepsilon}))
$$

Assumptions: $y \cdot \boldsymbol{b} < -\alpha|y|$ when $|y| \geq R$, and $\boldsymbol{\varrho}\boldsymbol{\varrho}^{\top}$ bounded

Issues:

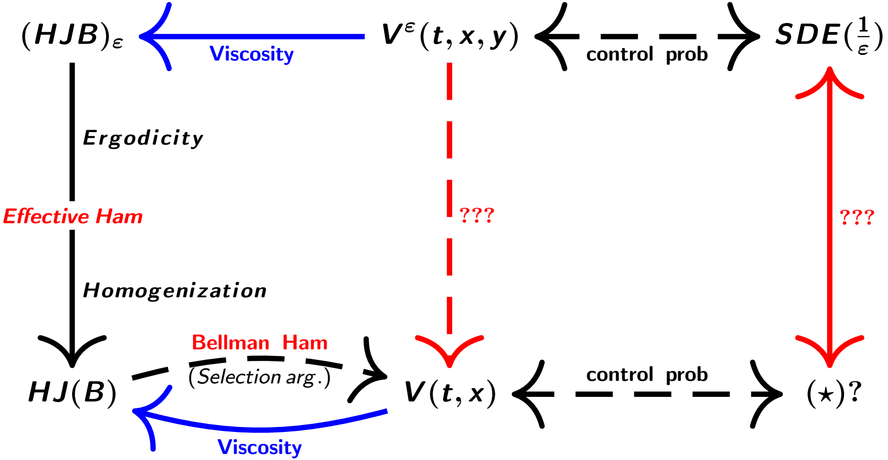
- High dimension : $\forall\, n, m \geq 1$
- Controlled dynamics : $\boldsymbol{u_t}$
- Unbounded domain : $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$
- Unbounded data:
  $|f|, \|\sigma\|, |b| \leq C(1 + |x| + |y|)$
- Possible degeneracy of σ and also ϱ

$\Longrightarrow$ Can we do something?

Yes, but...

[1] Ref.: **Bardi, M., & Cesaroni, A. (2011)**, and the references therein!

Plan


$(HJB)_\varepsilon$
Viscosity
$V^\varepsilon(t,x,y)$
control prob
$SDE(\frac{1}{\varepsilon})$
Ergodicity
Effective Ham
Homogenization
???
???
Bellman Ham
(Selection arg.)
$HJ(B)$
$V(t,x)$
control prob
$(\star)?$
Viscosity

Stochastic Control Problem with Singular Perturbations

$(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ a complete filtered probability space,
$(W_t)_t$ an $\mathcal{F}_t$-adapted standard r-dimensional Brownian motion,

$$
\begin{aligned}
dX_t &= \boldsymbol{f}(X_t, Y_t, \boldsymbol{u_t})\, dt + \sqrt{2}\boldsymbol{\sigma}^{\varepsilon}(X_t, Y_t, \boldsymbol{u_t})\, dW_t, \quad && X_0 = x \in \mathbb{R}^n \\
dY_t &= \frac{1}{\varepsilon}\boldsymbol{b}(X_t, Y_t)\, dt + \sqrt{\frac{2}{\varepsilon}}\boldsymbol{\varrho}(X_t, Y_t)\, dW_t, && Y_0 = y \in \mathbb{R}^m
\end{aligned} \tag{1}
$$

Pay-off function $\boldsymbol{J} : [0, T] \ni (t, x, y, u) \times \mathbb{R}^n \times \mathbb{R}^m \times \mathcal{U} \to \mathbb{R}, \;\; \lambda > 0$

$$
\boldsymbol{J}(t, x, y, u) := \mathbb{E}_{x,y}\left[e^{\lambda(t-T)}\boldsymbol{g}(X_T) + \int_t^T \ell(X_s, Y_s, u_s) e^{\lambda(s-T)} ds \right],
$$

Value function

$$
\boldsymbol{V}^{\varepsilon}(t, x, y) := \sup_{u \in \mathcal{U}} \{ \boldsymbol{J}(t, x, y, u)\ ,\ s.t.\ (X_., Y_.)\ \text{in}\ (1)\ \} \tag{2}
$$

$\boldsymbol{\mathcal{U}}$ the set of $\mathcal{F}_t$-progressively measurable processes taking values in U.

HJB equation

A fully nonlinear degenerate parabolic equation in $(\mathbf{0}, \boldsymbol{T}) \times \mathbb{R}^{\boldsymbol{n}} \times \mathbb{R}^{\boldsymbol{m}}$

$$\begin{cases} -\boldsymbol{V}_{\boldsymbol{t}}^{\varepsilon} + \boldsymbol{F}^{\varepsilon}\left(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{V}^{\varepsilon}, \boldsymbol{D_x V}^{\varepsilon}, \dfrac{\boldsymbol{D_y V}^{\varepsilon}}{\varepsilon}, \boldsymbol{D}^2_{\boldsymbol{xx}}\boldsymbol{V}^{\varepsilon}, \dfrac{\boldsymbol{D}^2_{\boldsymbol{yy}}\boldsymbol{V}^{\varepsilon}}{\varepsilon}, \dfrac{\boldsymbol{D}^2_{\boldsymbol{xy}}\boldsymbol{V}^{\varepsilon}}{\sqrt{\varepsilon}}\right) = \mathbf{0}, \\ \boldsymbol{V}^{\varepsilon}(\boldsymbol{T}, \boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{g}(\boldsymbol{x}), \quad \text{in } \mathbb{R}^{\boldsymbol{n}} \end{cases}$$

The Hamiltonian $F^{\varepsilon} : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{S}^n \times \mathbb{S}^m \times \mathbb{M}^{n,m} \to \mathbb{R}$ is

$$\boldsymbol{F}^{\varepsilon}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{s}, \boldsymbol{p}, \boldsymbol{q}, \boldsymbol{M}, \boldsymbol{N}, \boldsymbol{Z}) := \boldsymbol{H}^{\varepsilon}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{p}, \boldsymbol{M}, \boldsymbol{Z}) - \boldsymbol{\mathcal{L}}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{q}, \boldsymbol{N}) + \boldsymbol{\lambda s},$$

where

$$H^{\varepsilon}(x, y, p, M, Z) := \min_{u \in U}\left\{-\mathrm{tr}(\boldsymbol{\sigma}^{\varepsilon}\boldsymbol{\sigma}^{\varepsilon\top} M) - \boldsymbol{f} \cdot p - 2\mathrm{tr}(\boldsymbol{\sigma}^{\varepsilon}\boldsymbol{\varrho}^{\top} Z^{\top}) - \ell\right\}$$

$$\mathcal{L}(x, y, q, N) := \boldsymbol{b} \cdot q + \mathrm{tr}(\boldsymbol{\varrho}\boldsymbol{\varrho}^{\top} N)$$

$\boldsymbol{\sigma}^{\varepsilon}, \boldsymbol{f}, \boldsymbol{b}$ and $\boldsymbol{\ell}$ are computed at (x, y, u) and $\boldsymbol{\varrho} = \boldsymbol{\varrho}(x, y)$

Ergodicity

$dY_y(\cdot) = \boldsymbol{b}(x, Y_y(\cdot))\,dt + \sqrt{2}\varrho(x, Y_y(\cdot))\,dW_t,\;\; Y_y(0) = y \in \mathbb{R}^m,\;\; x$ fixed

It is well known[2] that an **invariant measure** $\boldsymbol{\mu_x}$ of $\boldsymbol{Y_y}(\cdot)$

- exists, is unique, has finite moments and Lip. Cont. density[3] w.r.t. x
- satisfies $\;\| \mathbb{P}_{Y_y(t)}(\cdot) - \boldsymbol{\mu_x}(\cdot) \|_{TV} \le C\,(1 + |y|^d)\,(1+t)^{-(1+k)}$

Moreover, we prove[4] for $\boldsymbol{\tau_n} := \inf\{t \ge 0 \text{ s.t. } \|Y_y(t)\| \ge n\}$,

Lemma

$$\exists\, \boldsymbol{\eta} > 0,\ \forall\, \beta > 0,\ \mathbb{E}\left[exp\left(-\frac{\boldsymbol{\tau_n}}{n^\beta} \right) \right] \le C\, n^\beta\, e^{-n\boldsymbol{\eta}} \xrightarrow[n \to +\infty]{} 0,\ (loc.\, unif.\, y)$$

[2] Veretennikov, *"On polynomial mixing and convergence rate for stochastic difference and differential equations."* Theory of Probability & Its Applications 44.2 (2000)

[3] Pardoux & Veretennikov, *"On Poisson equation and diffusion approximation 2."* The Annals of Probability (2003)

[4] In the line of proof [Prop.1.4] in: Herrmann, Imkeller, Peithmann, *"Transition times and stochastic resonance for multidimensional diffusions with time periodic drift: A large deviations approach"*, Ann. Appl. Probab.(2006)

Construct an Effective Hamiltonian

$\rightsquigarrow$ An **approximation** of the $\boldsymbol{\delta}$-Cell problem:

Let $\{D_n\}_n \subset \mathbb{R}^m$, ∂D_n smooth, $D_n \xrightarrow[n\to\infty]{} \mathbb{R}^m$ (e.g. D_n ball of radius n).

Consider the Dirichlet-Poisson problem, for $\boldsymbol{h}(\boldsymbol{y}) := \boldsymbol{H}(\overline{x}, \boldsymbol{y}, \overline{p}, \overline{M}, 0)$

$$\begin{cases} \boldsymbol{\delta\omega}(y) - \mathcal{L}\boldsymbol{\omega}(y) = -h(y), \text{ in } D_n \\ \boldsymbol{\omega}(y) = 0, \text{ on } \partial D_n \end{cases}$$

It has a unique solution $\boldsymbol{\omega}^{\boldsymbol{\delta},\boldsymbol{n}}(y) = \mathbb{E}\left[-\int_0^{\tau_n} h(Y_y(t))e^{-\delta t}dt\right]$
where τ_n is the first exist time of $Y_y(\cdot)$ from D_n.

Proposition

Let $\delta(n) = O\left(n^{-(4+\alpha)}\right)$, *for some* $\alpha > 0$, *the one has*

$$\lim_{n\to\infty}\left|\boldsymbol{\delta}(\boldsymbol{n})\boldsymbol{\omega}^{\boldsymbol{\delta}(\boldsymbol{n}),\boldsymbol{n}}(y) - \left(-\int_{\mathbb{R}^m} h(y)d\boldsymbol{\mu}(y)\right)\right| = 0, \quad \textit{loc. unif. in } y,$$

where $\boldsymbol{\mu}$ *is the unique invariant probability measure for the process* $Y_y(\cdot)$.

Convergence of the value function

The *effective* Hamiltonian is

$$\overline{H}(\overline{x}, \overline{p}, \overline{M}) := \int_{\mathbb{R}^m} H(\overline{x}, y, \overline{p}, \overline{M}, 0) d\mu(y)$$

The *effective* $\widetilde{HJB}$ equation is

$$\begin{cases} -V_t + \overline{H}(x, D_x V, D^2_{xx} V) + \lambda V(x) = 0, & (t, x) \in (0, T) \times \mathbb{R}^n \\ V(T, x) = g(x), & \text{in } \mathbb{R}^n \end{cases}$$

Theorem

The solution V^ε to $(HJB)_\varepsilon$ converges uniformly on compact subsets of $[0, T) \times \mathbb{R}^n \times \mathbb{R}^m$ to the unique continuous viscosity solution to the limit problem $\widetilde{HJB}$ satisfying a quadratic growth condition in x, i.e.

$$\exists\, K > 0 \text{ such that } |V(t, x)| \le K(1 + |x|^2), \quad \forall\, (t, x) \in [0, T] \times \mathbb{R}^n$$

Control representation of $\widetilde{HJB}$

Proposition

Under the standing assumptions, the effective Hamiltonian writes

$$\overline{\boldsymbol{H}}(\boldsymbol{x},\boldsymbol{p},\boldsymbol{M}) = \min_{\nu\in\mathcal{U}^{ex}(\boldsymbol{x})} \int_{\mathbb{R}^m} \left[-trace(\boldsymbol{\sigma}\boldsymbol{\sigma}^\top \boldsymbol{M}) - \boldsymbol{f}\cdot\boldsymbol{p} - \ell\right] \, \boldsymbol{d}\boldsymbol{\mu_x}(\boldsymbol{y})$$

where σ, f and ℓ are computed at (x,y,u), and $\mathcal{U}^{ex}(x)$ is the set of progressively measurable processes taking values in the extended control set $U^{ex}(x) := L^2((\mathbb{R}^m, \boldsymbol{\mu_x}), U)$.

The *extended* controls are

$$\nu_\cdot(\cdot) : t \mapsto \nu_t(\cdot) \in L^2((\mathbb{R}^m, \boldsymbol{\mu_{\hat{X}_t}}), U)$$
$$= \left\{ \phi(\cdot) : y \mapsto \phi(y) \in U \;\middle|\; \int_{\mathbb{R}^m} |\phi(y)|^2 \, d\boldsymbol{\mu_{\hat{X}_t}}(y) < \infty \right\}$$

$\rightsquigarrow$ This is an **exchange** operation " **min** $\int = \int$ **min**"

Limit Control Problem (I)

A guess for the limit dynamics:

$$\begin{cases} d\hat{X}_t = \int_{\mathbb{R}^m} f(\hat{X}_t, y, \boldsymbol{\nu}_t(y)) d\boldsymbol{\mu}_{\hat{\mathbf{X}}_t}(y)(y) dt \\ \qquad\qquad + \sqrt{2}\sqrt{\int_{\mathbb{R}^m} \sigma\sigma^\top(\hat{X}_t, y, \boldsymbol{\nu}_t(y)) d\boldsymbol{\mu}_{\hat{\mathbf{X}}_t}(y)}\, dW_t, \\ \boldsymbol{\nu}_t(\cdot) \in \mathcal{U}^{ex}(\hat{X}_t), \quad \text{and} \quad \hat{X}_0 = x \in \mathbb{R}^n. \end{cases} \tag{3}$$

The *effective* optimal control problem

$$V(t,x) = \sup \{ \hat{J}(t,x,\boldsymbol{\nu}_\cdot(\cdot)), \quad \text{subject to } (3) \} \tag{4}$$

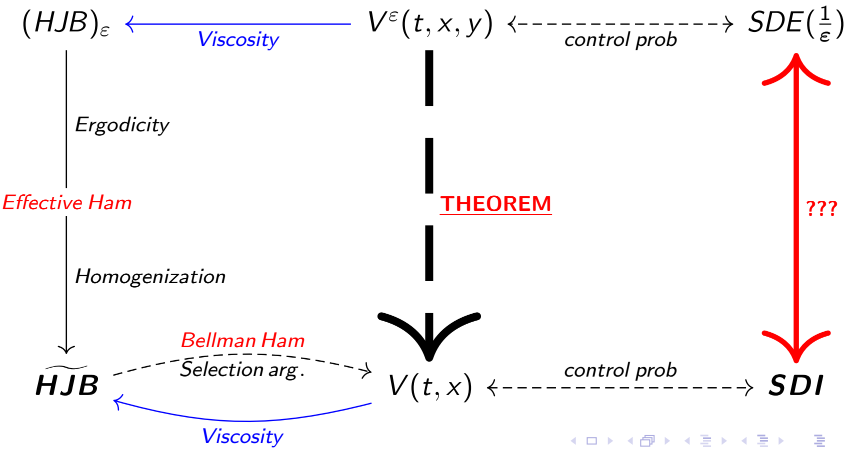
where the *effective* pay off $\hat{J}(t,x,\nu_\cdot(\cdot))$ is

$$\mathbb{E}_x\left[e^{\lambda(t-T)} g(\hat{X}_T) + \int_t^T \int_{\mathbb{R}^m} \ell(\hat{X}_s, y, \boldsymbol{\nu}_s(y)) d\boldsymbol{\mu}_{\hat{\mathbf{X}}_s}(y) e^{\lambda(s-T)} ds \right]$$

Limit Control Problem (II)

Theorem

The value function (4) is the unique viscosity solution to the Cauchy problem $\widetilde{HJB}$. It is in particular, the limit of V^ε defined in (2) for $(HJB)_\varepsilon$.



Convergence of Trajectories

Key observation:

- The *convergence Theorem* for the value function holds **independently** of the choice of the cost functional, i.e.

  $\rightsquigarrow$ As $\varepsilon \to 0$,

  $SDE(\frac{1}{\varepsilon})$ and SDI always produce the **same value** for **every choice** of a cost functional in the optimal control problem.

So we can *hope* for *at least* a convergence of the type

$$\lim_{\varepsilon \to 0} \max_{t \in [0,T]} \|\phi(X_t^{\varepsilon}) - \phi(\hat{\boldsymbol{X}}_t)\| = 0$$

where ϕ is any real valued continuous function.

Work in Progress

Application

Control of { Smoluchowski equation // Stochastic Gradient Descent[5] }

Let $\boldsymbol{V}$ be a confining potential. Let $\beta, \gamma > 0$ and $\sigma \geq 0$.

$$\begin{aligned} dX_t &= -u_t \underbrace{\gamma^{-1}(X_t - Y_t)}_{\nabla_x F(x,y)} dt + u_t \sqrt{2}\sigma\, dW_t, \quad X_0 = x \in \mathbb{R}^n \\ dY_t &= -\frac{1}{\varepsilon} \nabla_y F(X_t, Y_t) dt + \sqrt{\frac{2}{\varepsilon}} \beta^{-1/2}\, dW_t, \quad Y_0 = y \in \mathbb{R}^n \end{aligned} \tag{5}$$

where $F(x,y) := \boldsymbol{V}(y) + \frac{1}{2\gamma}|x-y|^2$. Therefore as $\varepsilon \to 0$ one expects

$$d\hat{X}_t = -\nu_t \nabla \boldsymbol{V}_\gamma(\hat{X}_t)\, dt + \nu_t \sqrt{2}\sigma\, dW_t \tag{6}$$

where $\boldsymbol{V}_\gamma$ is the local entropy, and β is the inverse temperature

$$\boldsymbol{V}_\gamma := -\frac{1}{\beta} \log\left(G_{\beta^{-1}\gamma} * \exp(-\beta \boldsymbol{V}) \right).$$

[5] **Highly inspired & motivated by:** Chaudhari, Oberman, Osher, Soatto & Carlier (2018). *Deep relaxation: partial differential equations for optimizing deep neural networks*